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Subharmonic response of a single-degree-of-freedom nonlinear vibroimpact system to a randomly disordered periodic excitation

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ABSTRACT

The subharmonic resonant response of a single-degree-of-freedom nonlinear vibroimpact oscillator with a one-sided barrier to narrow-band random excitation is investigated. The analysis is based on a special Zhuravlev transformation, which reduces the system to one without impacts or velocity jumps, thereby permitting the applications of asymptotic averaging over the period for slowly varying random processes. The averaged equations are solved exactly and algebraic equation of the amplitude of the response is obtained in the case without random disorder. A perturbation-based moment closure scheme is proposed and an iterative calculation equation for the mean square response amplitude is derived in the case with random disorder. The effects of damping, nonlinear intensity, detuning, and magnitudes of random excitations are analyzed. The theoretical analyses are verified by numerical results. Theoretical analyses and numerical simulations show that the peak amplitudes may be strongly reduced at large detuning or large nonlinear stiffness, and when intensity of the random disorder increase, the steady-state solution may change from a limit cycle to a diffused limit cycle, and even change to a chaos one.

1. Introduction

An impact oscillator, often named vibroimpact system, is the term used to represent a system, which is driven in some way and which also undergoes intermittent or a continuous sequence of contacts with motion limiting constraints [1]. Analyses of impact systems may be important in various engineering applications. Certain useful applications of vibration are known where impacts are involved, such as vibratory pile drivers, tie placers, etc. Analyses of impact motions may be important in the proper design of the corresponding machines and devices [2]. However, it is very difficult to investigate those systems. The main difficulty is that the dynamics of such systems are not continuous, but rather of intermittent type. In practice, engineering structures are often subjected to time dependent loadings of both deterministic and stochastic nature, such as the natural phenomena due to wind gusts, earthquakes, ocean waves, and random disturbance or noise which always exists in a physical system. The influence of random disturbance on the dynamical behavior of an impact dynamical system has caught the attention of many researchers. Some analyses methods, e.g. linearization method [3], quasi-static approach method [4,5], Markov processes method [6,7], stochastic averaging method [8–10], variable transformation method [11,12], energy balance method [13], mean impact Poincaré map method [14], and numerical simulation method [15] have been developed. In Ref. [2], the authors have tried to review and summarize the existing

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methods, results and literatures available for solving problem of stochastic vibroimpact systems. However, most of researches are focused on responses of linear impact oscillator (here, "linear" means that the differential equation of motion between impacts is linear) under wide-band random excitations and few are focused on the responses of linear [16] or nonlinear impact oscillator under narrow-band random excitation.

In this paper, the subharmonic response of a single-degree-of-freedom nonlinear vibroimpact oscillator with a one-sided barrier to narrow-band random excitation is investigated. The impact considered here is an instantaneous impact with restitution factor e. The paper is organized as follows. In Section 2, the Zhuravlev transformation and stochastic averaging method are used to obtain the mean square amplitude of the response. In Section 3, the directly numerical simulations verify the analytical result. Conclusions are presented in Section 4.

2. System description and theoretical analyses

Considering a single-degree-of-freedom nonlinear vibroimpact oscillator to random excitations:

$$\begin{cases} \ddot{y} + 2\beta\dot{y} + y + \alpha y^3 = f(t), & y > -\Delta \\ \dot{y}_+ = -e\dot{y}_-, & y = -\Delta \end{cases}, \tag{1}$$

where dot indicates differentiation with respect to time t, β is the damping coefficient, α represents the intensity of the nonlinear term, $0 < e \leq 1$ is the restitution factor to be a known parameter of impact losses, whereas subscripts "minus" and "plus" refer to value of response velocity just before and after the instantaneous impact. Thus $\dot{y}_+$ and $\dot{y}_-$ are actually rebound and impact velocities of the mass, respectively. They have the same magnitude whenever $e = 1$, therefore this special case is that of elastic impacts, whereas in case $e < 1$ some impact losses are observed, and $f(t)$ is a random process governed by the following equation [17]:

$$f(t) = h \sin \varphi(t), \quad \dot{\varphi}(t) = \Omega + \gamma\xi(t), \tag{2}$$

where $h > 0$ and $\Omega > 0$ are the amplitude and frequency of the random excitation, respectively, and $\xi(t)$ is a stationary Gaussian white noise of unit intensity, which describes random temporal deviations of the excitation frequency from its expected or mean Ω. The process $f(t)$ has the following power spectral density [15]:

$$S_f(\omega) = \frac{1}{4\pi} \frac{h^2\gamma^2(\Omega^2 + \omega^2 + \gamma^4/4)}{(\Omega^2 - \omega^2 + \gamma^4/4)^2 + \Omega^2\gamma^4}. \tag{3}$$

This process will be assumed to be narrow-band, which is clearly seen to be in the case provided that $\gamma \to 0$, and it is assumed that $\gamma \ll \Omega$ in this paper. The expression (3) is of the same form as one for the power spectral density of the narrow-band filtered Gaussian white noise.

Following Zhuravlev [18], the non-smooth transformation of state variables is introduced as follows:

$$y = |x| - \Delta, \quad \dot{y} = \dot{x}\,\mathrm{sgn}\,x, \tag{4}$$

where $\mathrm{sgn}\,x$ is the signum function such that $\mathrm{sgn}\,x = 1$ for $x > 0$ and $\mathrm{sgn}\,x = -1$ for $x < 0$. Obviously, this transformation makes the transformed velocity $\dot{x}$ continuous at the impact instants (i.e. $x = 0$) in the special case of elastic impact (i.e. $e = 1$), thereby reducing the problem to one without velocity jumps. However, this is not the case with a general vibroimpact system with impact losses, the jump of the transformed velocity $\dot{x}$ becomes proportional to $1-e$ instead of $1+e$ for the jump of original velocity $\dot{y}$. This jump may be included in the transformed differential equation of motion by using the Dirac delta function $\delta(x)$. Since $x(t_*) = 0$ at the impact instant t_* and $\delta(t - t_*) = |\dot{x}|\delta(x)$, the impulsive term can be obtained as

$$(\dot{x}_+ - \dot{x}_-)\delta(t - t_*) = (1 - e)\dot{x}|\dot{x}|\delta(x),$$

the transformed equation of motion can be written by substituting (4) into Eq. (1) as

$$\ddot{x} + x = -2\beta\dot{x} + \Delta\,\mathrm{sgn}\,x - (1 - e)\dot{x}|\dot{x}|\delta(x) - \alpha(|x| - \Delta)^3\,\mathrm{sgn}\,x + h\,\mathrm{sgn}\,x \sin \varphi(t). \tag{5}$$

Thus, the original impact system (1) is reduced to the "common" vibration system (5) without impact. The term $(1 - e)\dot{x}|\dot{x}|\delta(x)$ on the right hand side of Eq. (5) describes the impact losses of system, which can be regarded as an impulsive damping term. The transformed Eq. (5) permits rigorous analytical study by the asymptotic method of averaging over the period, as long as coefficients α, β, Δ, h, and $1-e$ are all small and proportional to a small parameter. Moreover, only subharmonic resonant responses will be considered, i.e. frequency Ω of the random excitation is near the subharmonic resonant responses $2n$, $\Omega \approx 2n$, where n is an arbitrary positive integer. The detuning parameter μ is defined according to $\mu = \Omega - 2n$, μ is assumed to be small and proportional to a small parameter. Then the response of Eq. (5) can be approximately represented as

$$x = A(t) \sin \Phi(t), \quad \dot{x} = A(t) \cos \Phi(t). \tag{6}$$

By introducing a new slowly varying phase shift $\theta(t) = \varphi(t) - 2n\Phi(t)$, Eq. (5) can be transformed to the following pair of first-order equations:

$$\begin{cases} \dot{A} = \cos\Phi[-2\beta A\cos\Phi + \Delta \mathrm{sgn}(\sin\Phi) - (1-e)A\cos\Phi|A\cos\Phi|\delta(A\sin\Phi) \\ \quad -\alpha(|A\sin\Phi| - \Delta)^3\mathrm{sgn}(\sin\Phi) + h\mathrm{sgn}(\sin\Phi)\sin(\theta + 2n\Phi)], \\ \dot{\theta} = \mu + \gamma\xi(t) + \frac{2n\sin\Phi}{A}[-2\beta A\cos\Phi + \Delta\mathrm{sgn}(\sin\Phi) - (1-e)A\cos\Phi|A\cos\Phi|\delta(A\sin\Phi) \\ \quad -\alpha(|A\sin\Phi| - \Delta)^3\mathrm{sgn}(\sin\Phi) + h\mathrm{sgn}(\sin\Phi)\sin(\theta + 2n\Phi)]. \end{cases} \tag{7}$$

Under the foregoing assumption that damping, nonlinearity intensity, impact losses and excitation terms are small, the right hand sides of both Eq. (7) are proportional to a small parameter, then A and θ are two slowly varying random processes with respect to time t, and Φ is a fast varying random process. By averaging over the fast state variable Φ [19], the following shortened equations can be obtained:

$$\begin{cases} \dot{A} = -\left(\beta + \dfrac{1-e}{\pi}\right)A + q\cos\theta, \\ \dot{\theta} = (\mu - 3n\alpha\Delta^2) - \dfrac{q}{A}\sin\theta + \dfrac{\rho}{A} + n\alpha\left(\dfrac{8}{\pi}\Delta A - \dfrac{3}{4}A^2\right) + \gamma\xi(t), \\ q = \dfrac{4nh}{(4n^2-1)\pi}, \quad \rho = \dfrac{4n\Delta}{\pi}(1 + \alpha\Delta^2). \end{cases} \tag{8}$$

Eq. (8) show that the difference between elastic impact ($e = 1$) and inelastic impact ($e<1$) is that inelastic impact increase the damping of the system from β to $\beta + (1 - e/\pi)$.

Consider first steady-state response for the case of a perfect periodicity as $\gamma = 0$, Eq. (8) become

$$\begin{cases} \dot{A} = -\left(\beta + \dfrac{1-e}{\pi}\right)A + q\cos\theta, \\ \dot{\theta} = (\mu - 3n\alpha\Delta^2) - \dfrac{q}{A}\sin\theta + \dfrac{\rho}{A} + n\alpha\left(\dfrac{8}{\pi}\Delta A - \dfrac{3}{4}A^2\right). \end{cases} \tag{9}$$

The steady-state solutions of Eq. (9) can be found by putting $A = A_0$, $\theta = \theta_0$, and $\dot{A} = 0$, $\dot{\theta} = 0$, this leads to the following result:

$$\begin{cases} \left(\beta + \dfrac{1-e}{\pi}\right)A_0 = q\cos\theta_0, \\ \rho + (\mu - 3n\alpha\Delta^2)A_0 + n\alpha\left(\dfrac{8}{\pi}\Delta A_0^2 - \dfrac{3}{4}A_0^3\right) = q\sin\theta_0. \end{cases} \tag{10}$$

Squaring and adding Eq. (10) yields the frequency-response equation

$$\left(\beta + \frac{1-e}{\pi}\right)^2 A_0^2 + \left[\rho + (\mu - 3n\alpha\Delta^2)A_0 + n\alpha\left(\frac{8}{\pi}\Delta A_0^2 - \frac{3}{4}A_0^3\right)\right]^2 = q^2. \tag{11}$$

Eq. (11) can be solved numerically for given parameters of the system. Eq. (11) have more than one steady-state solutions in some parameter areas, the stability of these steady-state responses can be examined by introducing some perturbation terms as

$$A = A_0 + A_1, \quad \theta = \theta_0 + \theta_1, \tag{12}$$

where A_0 and θ_0 are governed by Eqs. (10) and (11), A_1 and θ_0 are perturbation terms. Substituting Eq. (12) into Eq. (9) and neglecting the nonlinear terms, one obtains the following linearization of the modulation Eq. (9) at A_0, θ_0

$$\begin{cases} \dot{A}_1 = -\left(\beta + \dfrac{1-e}{\pi}\right)A_1 - q\sin\theta_0\theta_1, \\ \dot{\theta}_1 = \left[\dfrac{q}{A_0^2}\sin\theta_0 - \dfrac{\rho}{A_0^2} + n\alpha\left(\dfrac{8}{\pi}\Delta - \dfrac{3}{2}A_0\right)\right]A_1 - \left(\beta + \dfrac{1-e}{\pi}\right)\theta_1. \end{cases} \tag{13}$$

The eigenvalues of the coefficient matrix of system (13) are

$$\lambda_{1,2} = -\left(\beta + \frac{1-e}{\pi}\right) \pm \sqrt{c}, \quad c = -\left[\frac{q}{A_0^2}\sin\theta_0 - \frac{\rho}{A_0^2} + n\alpha\left(\frac{8}{\pi}\Delta - \frac{3}{2}A_0\right)\right]q\sin\theta_0. \tag{14}$$

Therefore the necessary and sufficient condition of the stability of the steady-state solutions A_0 and θ_0 is that the real parts of the eigenvalues $\lambda_{1,2}$ are less than zero, i.e. $c \le 0$, or

$$\left(\beta + \frac{1-e}{\pi}\right)^2 > c, \quad c > 0. \tag{15}$$

Next, we determine the steady-state response of system (8) in the stochastic case as $\gamma \neq 0$. Introducing another new pair of state variables

$$u = A\cos\theta, \quad v = A\sin\theta. \tag{16}$$

Eq. (8) can be transformed to

$$\begin{cases} \dot{u} = -\left(\beta + \dfrac{1-e}{\pi}\right)u - (\mu - 3n\alpha\Delta^2)v - \dfrac{\rho v}{\sqrt{u^2+v^2}} \\ \quad -n\alpha v\left[\dfrac{8}{\pi}\Delta\sqrt{u^2+v^2} - \dfrac{3}{4}(u^2+v^2)\right] + q - \gamma v\xi(t), \\ \dot{v} = -\left(\beta + \dfrac{1-e}{\pi}\right)v + (\mu - 3n\alpha\Delta^2)u + \dfrac{\rho u}{\sqrt{u^2+v^2}} \\ \quad +n\alpha u\left[\dfrac{8}{\pi}\Delta\sqrt{u^2+v^2} - \dfrac{3}{4}(u^2+v^2)\right] + \gamma u\xi(t). \end{cases} \tag{17}$$

In generally, Eq. (17) are taken as Stratonovich stochastic differential equations, or physical equations, which should be transformed to Ito ones by adding Wong–Zakai [15] correction terms for convenience, then Eq. (17) can be converted to the Ito-type stochastic differential equations as follows:

$$\begin{cases} \mathrm{d}u = \left[-\left(\beta + \dfrac{1-e}{\pi} + \dfrac{\gamma^2}{2}\right)u - (\mu - 3n\alpha\Delta^2)v - \dfrac{\rho v}{\sqrt{u^2+v^2}} \quad - n\alpha v\left[\dfrac{8}{\pi}\Delta\sqrt{u^2+v^2} - \dfrac{3}{4}(u^2+v^2) + q\right]\right]\mathrm{d}t - \gamma v\,\mathrm{d}W(t), \\ \mathrm{d}v = \left[-\left(\beta + \dfrac{1-e}{\pi} + \dfrac{\gamma^2}{2}\right)v + (\mu - 3n\alpha\Delta^2)u + \dfrac{\rho u}{\sqrt{u^2+v^2}} \quad + n\alpha u\left[\dfrac{8}{\pi}\Delta\sqrt{u^2+v^2} - \dfrac{3}{4}(u^2+v^2) + q\right]\right]\mathrm{d}t + \gamma u\,\mathrm{d}W(t), \end{cases} \tag{18}$$

where $W(t)$ is an unit Wiener process.

It should be noted that an exact analytical study to system (18) seems impossible due to nonlinear nature. Thus, approximate solutions of the second-order moments of the subharmonic response are proposed. Denoting u_k and v_k be the values of u and v in the kth iterative calculation, and substituting all terms u^2+v^2 in the right hand side of Eq. (18) by the mean square amplitude $(A_k^*)^2 = EA_k^2 = E(u_k^2 + v_k^2)$ approximately, an iterative calculation equations of Eq. (18) can be obtained as follows:

$$\begin{cases} \mathrm{d}u_{k+1} = [-au_{k+1} - b_k v_{k+1} + q]\,\mathrm{d}t - \gamma v_{k+1}\,\mathrm{d}W(t), \\ \mathrm{d}v_{k+1} = [-av_{k+1} + b_k u_{k+1}]\,\mathrm{d}t + \gamma u_{k+1}\,\mathrm{d}W(t), \\ a = \beta + \dfrac{1-e}{\pi} + \dfrac{\gamma^2}{2}, \\ b_k = \mu - 3n\alpha\Delta^2 + \dfrac{\rho}{A_k^*} + n\alpha\left[\dfrac{8}{\pi}\Delta A_k^* - \dfrac{3}{4}(A_k^*)^2\right], \\ \quad k = 0, 1, 2, \ldots . \end{cases} \tag{19}$$

Eq. (19) are linear Ito equations, so by using the Ito rule the steady-state moments Eu_{k+1} and Ev_{k+1} can be obtained by the moment method [17]. For the steady-state moments, one has

$$\frac{\mathrm{d}Eu_{k+1}}{\mathrm{d}t} = \frac{\mathrm{d}Ev_{k+1}}{\mathrm{d}t} = 0.$$

Taking expectation on both sides of Eq. (19), one obtains

$$aEu_{k+1} + b_k Ev_{k+1} = q, \quad aEv_{k+1} - b_k Eu_{k+1} = 0. \tag{20}$$

Eq. (20) have the following solutions:

$$Eu_{k+1} = \frac{aq}{a^2 + b_k^2}, \quad Ev_{k+1} = \frac{ab_k}{a^2 + b_k^2}. \tag{21}$$

Although the procedure can be easily extended to predict response moments of any order, only mean square amplitude $(A_{k+1}^*)^2 = E(u_{k+1}^2 + v_{k+1}^2)$ will be considered here. From Eq. (19), one obtains

$$\frac{\mathrm{d}A_{k+1}^2}{\mathrm{d}t} = 2u_{k+1}\frac{\mathrm{d}u_{k+1}}{\mathrm{d}t} + 2v_{k+1}\frac{\mathrm{d}v_{k+1}}{\mathrm{d}t} = -2aA_{k+1}^2 + 2qu_{k+1}. \tag{22}$$

For the steady-state moments, one has $\mathrm{d}EA_{k+1}^2/\mathrm{d}t = 0$. Taking expectation on both sides of Eq. (22), one obtains

$$(A_{k+1}^*)^2 = EA_{k+1}^2 = \frac{qEu_{k+1}}{a} = \frac{q^2}{a^2 + b_k^2}. \tag{23}$$

The initial value A_0^* of the iterative calculation can be taken as A_0, which is governed by Eq. (11). This iteration scheme may be expected to converge at least for small values of γ, when the mean amplitude is close to the deterministic value as governed by Eq. (11). Actually, this scheme works well in the numerical simulation. Then, the approximation mean square amplitude of system (18) can be taken as

$$(A^*)^2 = \lim_{k\to\infty} (A_{k+1}^*)^2 = \lim_{k\to\infty} \frac{q^2}{a^2 + b_k^2}. \tag{24}$$

3. Numerical simulation

In this section, the analytical results will be shown and compared with the directly numerical results. All the directly numerical simulations by using Monte-Carlo method are based on the original system dominated by Eq. (1), and can give powerful validation with analytical results. For the method of numerical simulation, readers can refer to Zhu [20] and Shinozuka [21,22]. In this paper, the power spectrum of $\xi(t)$ is taken as

$$S(\omega) = \begin{cases} \dfrac{1}{2\pi}, & 0<\omega \le 2\Omega \\ 0, & \omega > 2\Omega \end{cases}.$$

For numerical simulation it is more convenient to use the pseudorandom signal given by [20]

$$\xi(t) = \sqrt{\frac{2\Omega}{N\pi}} \sum_{k=1}^{N} \cos\left[\frac{\Omega}{N}(2k-1)t + \varphi_k\right],$$

where φ_k's are independent and uniformly distributed in $(0, 2\pi]$, and N is a larger integer number.

Monte-Carlo simulations are focused on the first-order subharmonics $(n = 1, \Omega \approx 2)$, although the higher-order subharmonics $(\Omega \approx 2n,\ n = 2, 3, 4, \ldots)$ simulations should be of the same importance. In the numerical simulation, the parameters in system (1) are chosen as $h = 0.5$, $e = 0.9$, $\Delta = 0.05$, $n = 1$. The governing Eq. (1) is numerically integrated by the fourth-order Runge–Kutta algorithm between impacts, which is valid until the first encounter with the barriers, that is until the equality $y = \Delta$ is satisfied. The impact condition $\dot{y}_+ = -e\dot{y}_-$ is then imposed, using the numerical solution $\dot{y}_-$. This results in the rebound velocity $\dot{y}_+$, thereby providing the initial values for the next step numerical calculation. The numerical results are shown from Figs. 1–6.

We first consider the perfectly periodic excitation in the case $\gamma = 0$, $\alpha = 0.05$, $\beta = 0.15$, and $\alpha = 0.05$, $\beta = 0.1$. The variations of the steady-state response A_0 with Ω are shown in Fig. 1, for comparison the theoretical results given by Eq. (11) are also shown in Fig. 1. From Eq. (6), one has $A^2 = x^2 + \dot{x}^2 = (y + \Delta)^2 + \dot{y}^2$, therefore the mean square response amplitude was calculated as $A_0^2 = A_*^2 = \langle (y+\Delta)^2 \rangle + \langle (\dot{y})^2 \rangle$ in numerical simulation, where angular brackets denote common time averaging for the response sample. Fig. 1 shows that the deterministic response predicted by the averaging method is in good agreement with that obtained by numerical simulations.

It can be seen from Fig. 1 that the response amplitude will decrease when the damping β increases, which is in accordance with the physical intuition. The peak response amplitude will become large when the frequency Ω is near the

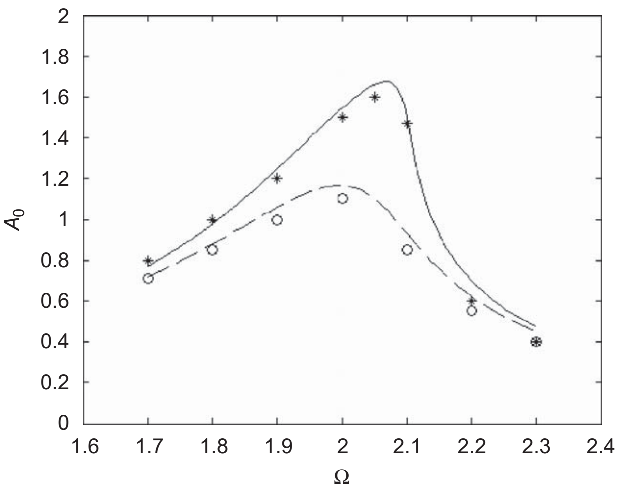

Fig. 1. Frequency response of system (1) $\alpha = 0.05$, $\gamma = 0$: —— theoretical solution ($\boldsymbol{\beta} = 0.1$), – – – theoretical solution ($\boldsymbol{\beta} = 0.15$), *** numerical solution ($\boldsymbol{\beta} = 0.1$), and ∘∘∘ numerical solution ($\boldsymbol{\beta} = 0.15$).

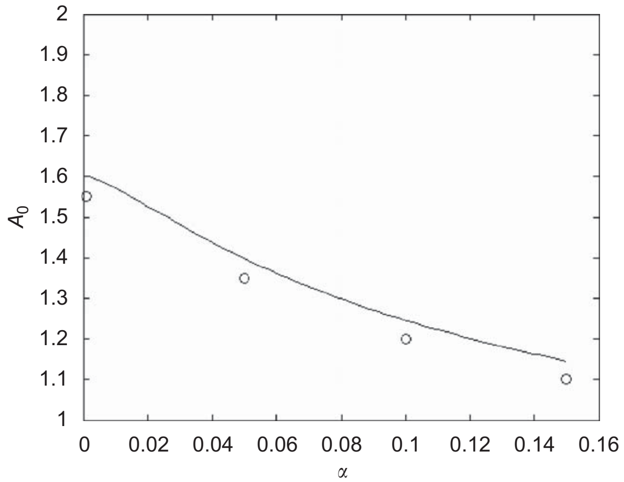

Fig. 2. Frequency response of system (1) ($\Omega = 1.95$, $\beta = 0.1$, $\gamma = 0.0$): —— theoretical solution and ∘∘∘ numerical solution.

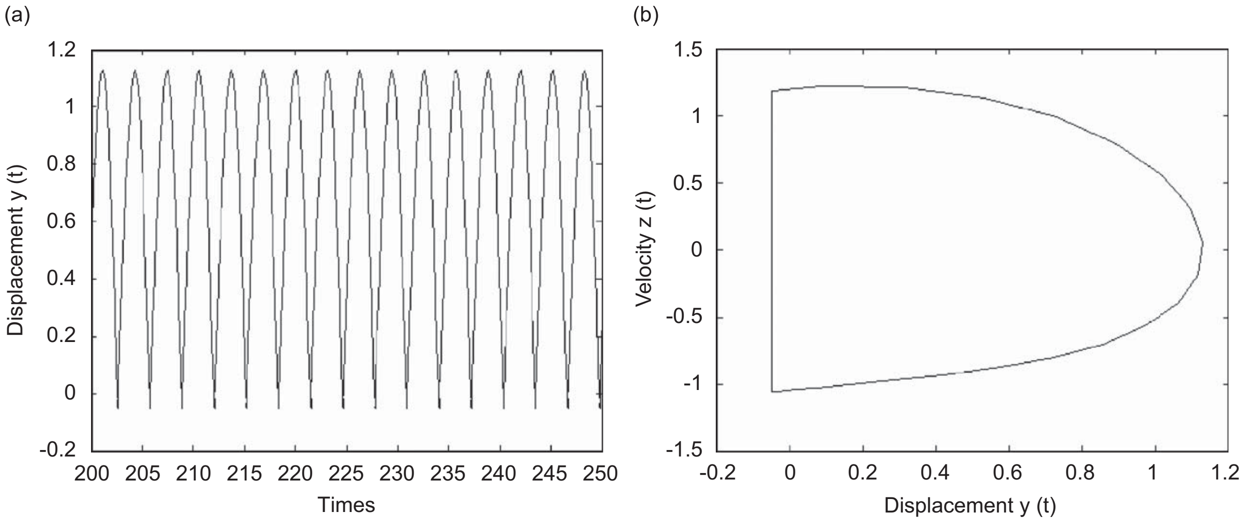

Fig. 3. Numerical results of Eq. (1) ($\gamma = 0$, $\alpha = 0.05$, $\beta = 0.15$, $\Omega = 2.0$): (a) time history of $y(t)$ and (b)phase plot.

resonant frequency $\Omega = 2$, and will decrease strongly when Ω departs from the resonant frequency. Comparing with the numerical solution, the accuracy of the analytical solution is seen to be reduced a little in the case of large detuning, this may be partly due to some inaccuracy of the Krylov–Bogoliubov averaging method at large detuning.

Now we consider the effect of the intensity of nonlinearity α on the response amplitude A_0 of the system. The variations of the steady-state response A_0 with α are shown in Fig. 2 for the case $\Omega = 1.95$, $\beta = 0.1$, $\gamma = 0.0$, for comparison the theoretical results given by Eq. (11) are also shown in Fig. 2. It can be seen from Fig. 2 that the response amplitude A_0 will decrease strongly when α increases, therefore the nonlinearity should be considered in the analysis of deterministic impact system.

The response time history of system (1) and the phase plot are shown in Fig. 3 in the case $\gamma = 0$, $\alpha = 0.05$, $\beta = 0.15$, $\Omega = 2.0$, where $z(t) = \dot{y}(t)$ denotes the velocity of the mass. Clearly, the response is a period one while the phase trajectory is a limit cycle.

Next, we determine the effect of the noise term $\xi(t)$ on the primary response. The variations of the steady-state response A^* with Ω in the case $\gamma = 0.25$, $\alpha = 0.05$, $\beta = 0.15$ and $\alpha = 0.05$, $\beta = 0.1$ are shown in Fig. 4, for comparison the theoretical results given by Eq. (24) are also shown in Fig. 4.

Similarly, high accuracy of the analytical method can also be claimed for the case under non-perfectly periodic excitation as can be seen from Fig. 4. Once again, strongly reduction of the peak response amplitude due to large damping and large detuning can be seen. Noting that the parameters of system (1) corresponding to Figs. 1 and 4 are the same except for γ, a rather drastic reduction of peak response amplitudes due to random disorder $\xi(t)$ in the excitation can be seen form

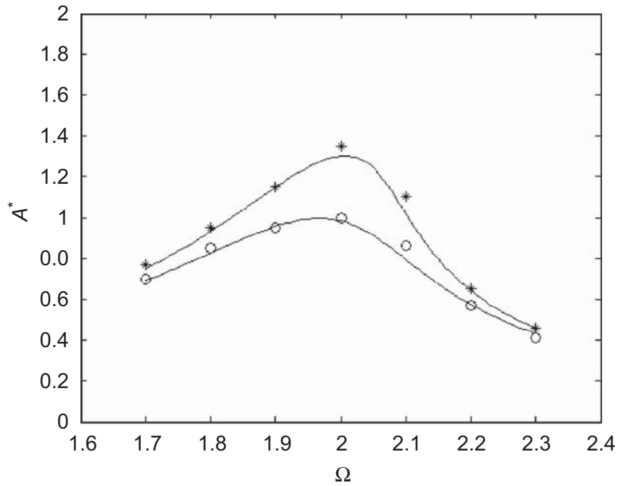

Fig. 4. Frequency response of system (1) $\alpha = 0.05$, $\gamma = 0.25$: —— theoretical solution ($\boldsymbol{\beta} = 0.1$), - - - theoretical solution ($\boldsymbol{\beta} = 0.15$), *** numerical solution ($\boldsymbol{\beta} = 0.1$), and ∘∘∘ numerical solution ($\boldsymbol{\beta} = 0.15$).

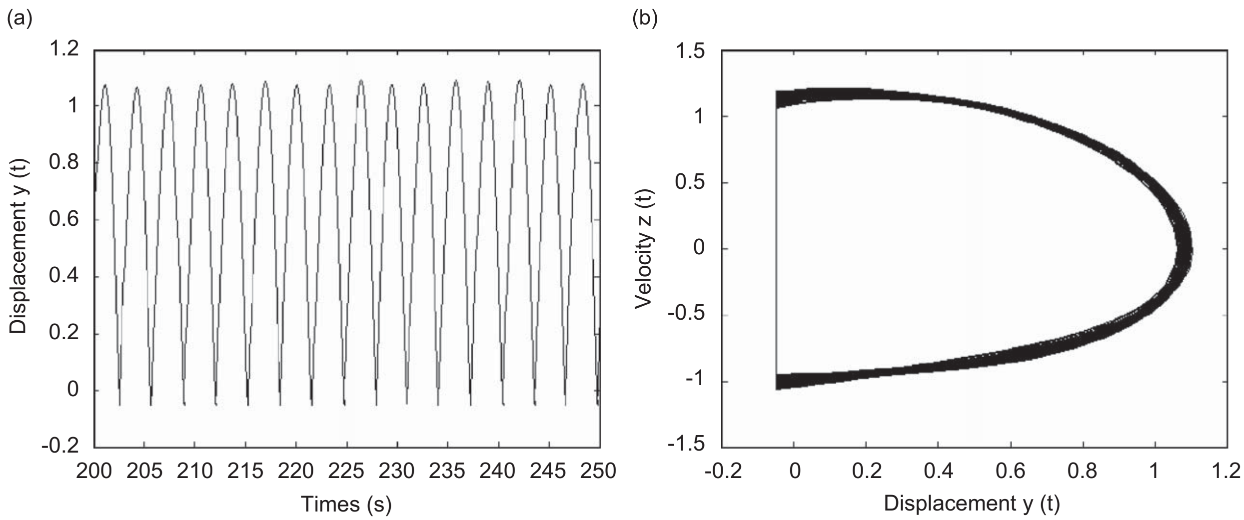

Fig. 5. Numerical results of Eq. (1) ($\gamma = 0.25$, $\alpha = 0.05$, $\beta = 0.15$, $\Omega = 2.0$): (a) time history of $y(t)$ and (b) phase plot.

Fig. 4 in comparison with Fig. 1. Such phenomena can also be illustrated in Fig. 5, where the parameters of system (1) are $\gamma = 0.25$, $\beta = 0.15, \Omega = 2.0$, which are the same corresponding to Fig. 3 except for γ.

In comparison with Fig. 3, it can be seen from Fig. 5 that random noise $\xi(t)$ will reduce the peak response amplitude, and change the steady-state response of system (1) from a periodic solution to a quasi-periodic one such that change the phase trajectory from a limit cycle to a diffused limit cycle. Further numerical simulations show that the width of the diffused limit cycle will be large when the intensity of the random disorder increases.

Now we consider the effect of the intensity of nonlinearity α on the response amplitude A_* of the system. The variations of the steady-state response A_* with α are shown in Fig. 6 for the case $\Omega = 1.95$, $\beta = 0.1$, $\gamma = 0.25$, for comparison the theoretical results given by Eq. (24) are also shown in Fig. 6. It can be seen from Fig. 6 that the response amplitude A_* will decrease strongly when α increases, therefore the nonlinearity should also be considered in the analysis of stochastic impact system.

Eq. (11) will have multiple real and positive stable solutions in some parameter domain, and then system (1) may have multiple stable steady-state responses. Such phenomena is found in the following numerical simulations, where the parameters of system (1) are

$$\alpha = 0.05,\ \beta = 0.25,\ h = 2.5,\ e = 0.8,\ \Delta = 0.35,\ \Omega = 2.45,\ n = 1.$$

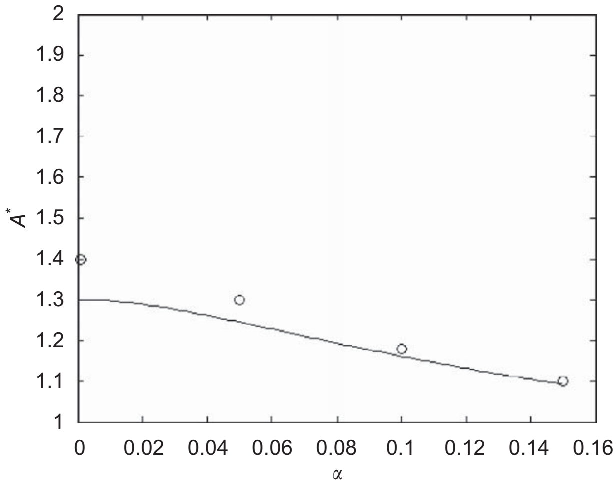

Fig. 6. Frequency response of system (1) ($\Omega = 1.95,\ \beta = 0.1,\ \gamma = 0.25$): —— theoretical solution and∘∘∘ numerical solution.

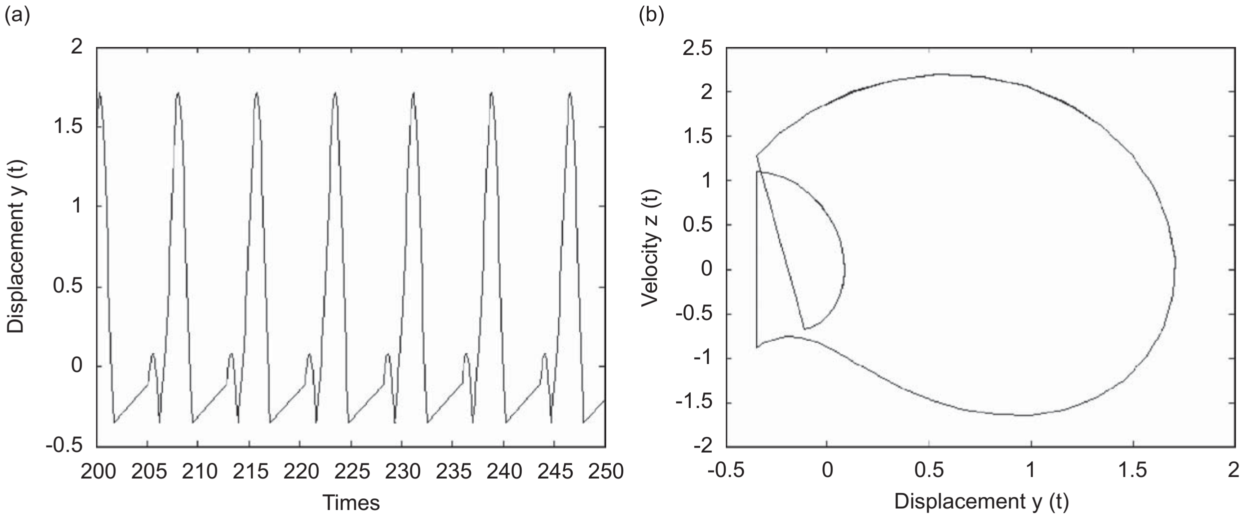

Fig. 7. Numerical results of Eq. (1) ($\gamma = 0.0$): (a) time history of $y(t)$ and (b) phase plot.

Eq. (11) has three real and positive solutions $A_0 = 1.3509, 3.0017, 5.5333$, and only the two solutions $A_0 = 1.3509, 3.0017$ are stable according to condition (15). The numerical results for system (1) are shown in Figs. 7–9 for different γ. The time history of the response and the phase trajectory are plotted in Fig. 7 in the case $\gamma = 0$.

The response of system (1) shown in Fig. 7 is a period-two one, and will become quasi-periodic ones when γ increase, as shown in Fig. 8 in case $\gamma = 0.004, 0.001$.

The phase trajectories shown in Fig. 8 represent the responses are similar to period-two solutions and have some pervasion, one may call them quasi-periodic-two solutions. The pervasion of the phase trajectory will strengthen as γ increases, and such pervasion will even destroy the topological property of the phase trajectory as shown in Fig. 9 in case $\gamma = 0.06$. The phase trajectories shown in Fig. 9 are different completely from which shown in Figs. 7 and 8, and one may call such response a stochastic chaos.

4. Conclusions and discussion

In this paper, the methods of Zhuravlev transformation and stochastic averaging are used to analyze the response of a nonlinear impact system under disordered periodic excitation. So far, exact solutions of nonlinear impact system under random excitation are only available for a very limited number of problems. Thus, approximate methods have been developed and used to treat many of these problems. These include the method of equivalent or stochastic linearization,

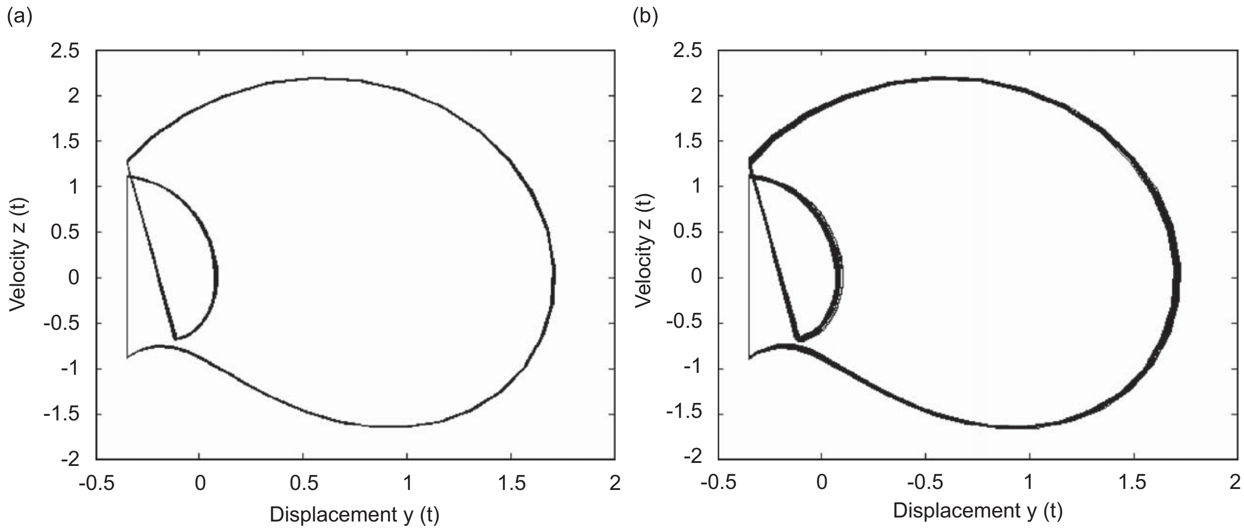

Fig. 8. Phase plot of system (1): (a) $\gamma = 0.004$ and (b) $\gamma = 0.01$.

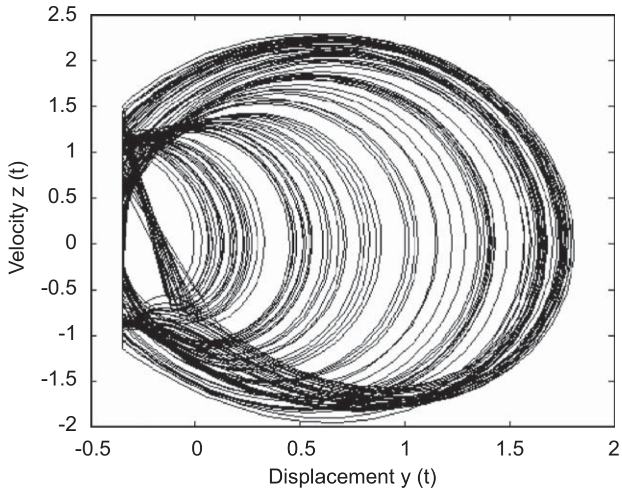

Fig. 9. Phase plot of system (1): $\gamma = 0.06$.

perturbation methods, stochastic averaging and series expansions, etc. In fact, even if in the single degree of freedom nonlinear deterministic system it is difficulty or impossible to solve exactly, hence approximate method have widely used in the analysis of determine nonlinear system. These include small parameter method, method of coordinate transformation, multiple scale method, method of slowly varying parameter, KBM method, method of equivalent linearization, method of harmonic balance, etc. The approximate methods in determine system can be extended to random system. For example, in recent year, Rajan and Davies [23], Nayfeh and Serhan [24] have extended the method of multiple scales to the analysis of nonlinear systems under random external excitations, and the authors of this paper [25,26] extended this method to the analysis of nonlinear systems under random parameter excitation. Harmonic balance method [27] and polynomial approximation method [28] have also been extended to random systems.

Theoretical analyses and numerical simulations show that the peak response amplitude will decrease when the intensity of the random disorder γ in the periodic excitation, damping β, and nonlinear intensity α increase. The random disorder $\xi(t)$ will change the steady-state response of system (1) from a periodic solution to a quasi-periodic one, and even to stochastic chaos.

The model in Eq. (2) may be less accurate in some applications, therefore other model such as filtered Gaussian white noise may be more appropriate for the basic narrow-band excitation, this will be continuation of this research.

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